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Stability for a class of semilinear fractional stochastic integral equations

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Abstract

In this paper we study some stability criteria for some semilinear integral equations with a function as initial condition and with additive noise, which is a Young integral that could be a functional of fractional Brownian motion. Namely, we consider stability in the mean, asymptotic stability, stability, global stability, and Mittag-Leffler stability. To do so, we use comparison results for fractional equations and an equation (in terms of Mittag-Leffler functions) whose family of solutions includes those of the underlying equation.

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1 Introduction

Currently fractional systems are of great interest because of the applications they have in several areas of science and technology, such as engineering, physics, chemistry, mechanics, *etc.* (see, *e.g.*, [1–4] and the references therein). Particularly we can mention system identification [1], robotics [5], control [1, 4], electromagnetic theory [6], chaotic dynamics and synchronization [7–10], applications on viscoelasticity [11], analysis of electrode processes [12], Lorenz systems [7], systems with retards [13], quantic evolution of complex systems [14], numerical methods for fractional partial differential equations [15–17], among other. A nice survey of basic properties of deterministic fractional differential equations is in Lakshmikantham and Vatsala [18]. Also, many researchers have established stability criteria of mild solutions of stochastic fractional differential equations using different techniques.

For deterministic systems, the stability of fractional linear equations has been analyzed by Matignon [19] and Radwan *et al.* [20]. Besides, several authors have studied nonlinear cases using Lyapunov method (see, *e.g.*, Li *et al.* [21] and references therein). In particular, nonlinear fractional systems with a function as initial condition using also the Lyapunov technique have been considered in the PhD thesis of Martínez-Martínez [22]. Moreover, in the work of Junsheng *et al.* [23] the form of the solution for a linear fractional equation with a constant initial condition in terms of the Mittag-Leffler function is given by means of the Adomian decomposition method. Wen *et al.* [24] have established stability results for fractional nonlinear equations via the Gronwall inequality. Equality (3) below can be seen as an extension of the results in [10] and the Gronwall inequality stated in [24]. In [24], the stability is used to obtain synchronization of fractional systems.

On the other hand, a process used frequently in the literature is fractional Brownian motion $B^H = \{B_t^H, t \geq 0\}$ due to the wide range of properties it has, such as long range memory (when the Hurst parameter H is greater than one half) and intermittency (when $H < 1/2$). Unfortunately, in general, it is not a semimartingale (the exception is $H = 1/2$). Thus, we cannot use classical Itô calculus in order to integrate processes with respect to B^H when $H \neq 1/2$, but we may use other approaches, such as Young integration (see Gubinelli [25], Young [26], Zähle [27], Dudley and Norvaiša [28], Lyons [29]). The reader may also refer to Nualart [30], and Russo and Vallois [31] for other types of integrals. As a consequence, an important application is the analysis of stochastic integral equations driven by fractional Brownian motion that has been considered by several authors these days for different interpretations of stochastic integrals (see, *e.g.*, Lyons [29], Quer-Sardanyons and Tindel [32], León and Tindel [33], Nualart [30], Friz and Hairer [34], Lin [35] and Nualart and Rǎşcanu [36]).

The stability of stochastic systems driven by Brownian motion has also been studied. Some authors use a fundamental solution of this equations in order to investigate the stability of random systems. An example of this is the paper of Applebay and Freeman [37], who gave the solution in terms of the principal matrix of integrodifferential equations with an Itô integral noise and find the equivalence between almost sure exponential convergence and the pth mean exponential convergence to zero for these systems. Bao [38] used the Gronwall inequality to state the mean square stability for Volterra-Itô equations with a function as initial condition and bounded kernels. Several researchers have studied stability of stochastic systems via Lyapunov function techniques. An example of this is the paper of Li *et al.* [39], who prove stability in probability for Itô-Volterra integral equation; also Zhang *et al.* [40] have stated a stochastic type stability criterion for stochastic integrodifferential equations with infinite retard, and Zhang and Zhang [41] have dealt with conditional stability of Skorohod Volterra type equations with anticipative kernel. Nguyen [42] presented the solution via the fundamental solution for linear stochastic differential equations with time-varying delays to obtain the exponential stability of these systems. The noise is an additive one and has the form $\int_0^\cdot \sigma(s)\,dW_s^H$. Here

$$W_t^H = \int_0^t (t-s)^{H-1/2}\,dW_s, \quad H \in (1/2, 1),$$

W is a Brownian motion and σ is a deterministic function such that

$$\int_0^\infty \sigma^2(s) e^{2\lambda s}\,ds < \infty,$$

for some $\lambda > 0$. Also, Zeng *et al.* [43] utilize the Lyapunov function techniques to prove stability in probability and moment exponential stability for stochastic differential equation driven by fractional Brownian motion with parameter $H > 1/2$. Yan and Zhang [44] proved sufficient conditions for the asymptotical stability in the pth moment for the closed form of the solution to a fractional impulsive partial neutral stochastic integrodifferential equation with state dependent retard in Hilbert space. In the linear case, Fiel *et al.* [45] have used the Adomian decomposition method to find the mild solution of a stochastic fractional integral equation with a function as initial condition driven by a Hölder continuous process in terms of Young or Skorohod integrals. This closed form is given in terms of the Mittag-Leffler functions. The stability in the large and stability in the mean sense of these random systems is also analyzed. As an application, the stability of equations driven by a functional of fractional Brownian motion is derived.

In this paper we extend the results given in [45] and [24], that is, we study the stability of the solution to the equation

$$X(t) = \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \bigl[AX(s) + h\bigl(X(s)\bigr)\bigr] \, ds + \mathcal{Z}_t, \quad t \geq 0. \tag{1}$$

The initial condition $\xi = \{\xi_t, t \geq 0\}$ is a function, h is a $O(x)$ as $x \to 0$ (*i.e.*, we have $C > 0$ and $\delta > 0$ such that $|h(x)| \leq C|x|$ for $|x| < \delta$), $\beta \in (0,1)$, $A < 0$, and $\mathcal{Z}$ is a Young integral of the form

$$\mathcal{Z}_t = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, d\theta_s.$$

Here, $\theta = \{\theta_s, s \geq 0\}$ is a γ-Hölder continuous function that may represent the paths of a functional of fractional Brownian motion, where $\gamma \in (0,1)$, $\alpha \in (1,2)$, and $\alpha + \gamma > 2$. Unlike other papers where the involved kernels are bounded functions we consider the case that kernels are not bounded, and we use comparison results as a main tool.

We observe that equation (1) can be a useful model for applications in several areas of science. For example, (1) provides a fractional version of the Verhlust-Pearl equation (see Scudo [46], p.5), where $h(x) = \lambda x^2$, for some $\lambda \in \mathbb{R}$. So, now the stability of a single insolated species can be analyzed by means of fractional systems. Also, as pointed out in [24], for some fractional systems in engineering h is a $O(x)$ as $x \to 0$.

This work is organized as follows. In Section 2 we introduce a fractional integral equation, whose family of solutions includes those of (1). Also, in Section 2, we state a comparison result for fractional systems that becomes the main tool for our results. In Section 3, we study some stability criteria for equation (1) in the case that $\mathcal{Z} \equiv 0$. These results can be seen as extensions of the results given in [45] and [24]. Finally, the stability of equation (1) in the case that θ is either a Hölder continuous process or a functional of fractional Brownian motion is considered in Section 4.

2 Preliminaries

In this section we introduce the framework and the definitions that we use to prove our results. Part of the main tool that we need is the stability of some fractional linear systems as presented by Fiel *et al.* [45] and a comparison result (see Lemma 1 below).

2.1 The Young integral

For $T > 0$ and $\gamma \in (0,1)$, let $C_1^\gamma([0,T];\mathbb{R})$ be the set of γ-Hölder continuous functions $g : [0,T] \to \mathbb{R}$ of one variable such that the seminorm

$$\|g\|_{\gamma,[0,T]} := \sup_{r,t \in [0,T], r \neq t} \frac{|g_t - g_r|}{|t-r|^\gamma},$$

is finite.

The Young integral was initially defined for functions with p-variation in the work of Young [26]. In particular, for $f \in C_1^\kappa([0,T];\mathbb{R})$ and $g \in C_1^\gamma([0,T];\mathbb{R})$, with $\kappa + \gamma > 1$ the Young integral is well defined, and for $s \le t \le T$ it is given by

$$\int_s^t f_u \, dg_u = \lim_{|\Pi_{st}| \to 0} \sum_{i=0}^{n-1} f_{t_i}(g_{t_{i+1}} - g_{t_i}),$$

where the limit is over any partition $\Pi_{st} = \{t_0 = s, \dots, t_n = t\}$ of $[s,t]$, whose mesh tends to zero.

We observe that this integral has been extended by Zähle [27], Gubinelli [25], Lyons [29], among others. For a detailed exposition on the Young integral the reader is referred to the paper of Dudley and Norvaiša [28] (see also Gubinelli [25], and León and Tindel [33]).

2.2 Semilinear Volterra integral equations with additive noise

Here we consider the Volterra integral equation

$$X(t) = \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} A X(s) \, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, d\theta_s, \quad t \ge 0, \tag{2}$$

where the initial condition $\xi = \{\xi_t, t \ge 0\}$ is bounded on compact sets and measurable, $\beta \in (0,1)$, $A \in \mathbb{R}$, $\alpha \in (1,2)$, $\theta = \{\theta_s, s \ge 0\}$ is a γ-Hölder continuous function with $\gamma \in (0,1)$ and Γ is the gamma function. The second integral in (2) is a Young one and it is well defined if $\alpha - 1 + \gamma > 1$, because $s \mapsto (t-s)^{\alpha-1}$ is $(\alpha-1)$-Hölder continuous on $[0,t]$.

Remember that, for $z \in \mathbb{R}$, the Mittag-Leffler function is defined as

$$E_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ka+b)}, \quad a, b > 0,$$

where Γ is the gamma function. In order to see a more detailed exposition of this function, the reader is referred to the book of Podlubny [4].

The closed form for the solution of equation (2) is

$$\begin{aligned} X(t) = {} & \xi_t + A \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(A(t-s)^\beta\big) \xi_s \, ds \\ & + \int_0^t (t-s)^{\alpha-1} E_{\beta,\alpha}\big(A(t-s)^\beta\big) \, d\theta_s, \quad t \ge 0. \end{aligned} \tag{3}$$

The reader can consult [45] for its proof.

In this work we use comparison methods in order to obtain the stability of some fractional systems. We can find comparison theorems in the literature for fractional evolution equations (see, *e.g.*, Theorem 4.2 in [18]), but, unfortunately these results are not suitable for our purpose. Thus, we give the following lemma, which is a version of Theorem 2.2.5 in Pachpatte [47] and allows us to prove stability for the semilinear equations that we study. Hence, this result is a fundamental tool in the development of this paper.

Lemma 1 *Let $k : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a function such that*:

(i) *$k(\cdot, x)$ is measurable on $[0,T]$ for each $x \in \mathbb{R}$*;

(ii) *There is a constant $M > 0$ such that $|k(s,x) - k(s,y)| \le M|x-y|$, for any $s \in [0,T]$ and $x, y \in \mathbb{R}$*;

(iii) *k is bounded on bounded sets of* $[0,T] \times \mathbb{R}$;

(iv) $k(s,\cdot)$ *is non-decreasing for any* $s \in [0,T]$.

Also, let $B \in \mathbb{R}$, $\beta \in (0,1)$, *and* x *and* y *two continuous functions on* $[0,T]$ *such that*

$$x(t) \le y(t) + \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(B(t-s)^\beta\big) k\big(s,x(s)\big)\,ds, \quad t \in [0,T]. \tag{4}$$

Then $x \le u$ *on* $[0,T]$, *where* u *is the solution to the equation*

$$u(t) = y(t) + \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(B(t-s)^\beta\big) k\big(s,u(s)\big)\,ds, \quad t \in [0,T]. \tag{5}$$

Remark The assumptions of k entail that equation (5) has a unique continuous solution.

Proof Denote by $C([0,T])$ the family of continuous functions on $[0,T]$. Let $\mathcal{G} : C([0,T]) \to C([0,T])$ be given by

$$(\mathcal{G}z)(t) = y(t) + \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(B(t-s)^\beta\big) k\big(s,z(s)\big)\,ds, \quad t \in [0,T].$$

It is not difficult to see that hypotheses (i) and (iii) imply that $\mathcal{G}$ is well defined. It means that $\mathcal{G}(z)$ is a continuous function for each $z \in C([0,T])$. We denote $\sup_{t\in[a,b]} |z(t)|$ by $\|z\|_{\infty,[a,b]}$ for any function $z \in C([a,b])$. Then, from the continuity of $E_{\beta,\beta}$ and hypothesis (ii), there is a constant $\bar{M} > 0$ such that, for every $z, \tilde{z} \in C([0,T])$, we have

$$\begin{aligned}
\big|(\mathcal{G}z)(t) - (\mathcal{G}\tilde{z})(t)\big| &\le \bar{M} \int_0^t (t-s)^{\beta-1} \big|k\big(s,z(s)\big) - k\big(s,\tilde{z}(s)\big)\big|\,ds \\
&\le M\bar{M} \int_0^t (t-s)^{\beta-1} \big|z(s) - \tilde{z}(s)\big|\,ds \\
&\le \frac{M\bar{M}}{\beta} T^\beta \|z - \tilde{z}\|_{\infty,[0,T]}, \quad \text{for } t \in [0,T].
\end{aligned}$$

Similarly, for $\bar{T} \le T$, we are able to see that

$$\|\mathcal{G}z - \mathcal{G}\tilde{z}\|_{\infty,[0,\bar{T}]} \le \frac{M\bar{M}}{\beta} \|z - \tilde{z}\|_{\infty,[0,\bar{T}]} \bar{T}^\beta.$$

Consequently, if $\bar{T}^\beta \frac{M\bar{M}}{\beta} < 1$, $\mathcal{G}$ is a contraction on $C([0,\bar{T}])$. Therefore the sequence $v_{n+1} = \mathcal{G}v_n$, with $v_0 = x$, is such that $v_n(t) \to u(t)$ and $v_n(t) \le v_{n+1}(t)$ for $t \in [0,\bar{T}]$, due to hypothesis (iv), (4), and $E_{\beta,\beta}$ being a completely monotonic function (see Miller *et al.* [48] or Schneider [49]). Thus the result is true if we write $\bar{T}$ instead of T.

Now, suppose the lemma holds for the interval $[0,n\bar{T}]$, $n \in \mathbb{N}$. Then, by (4) we can write

$$\begin{aligned}
x(t) &\le y(t) + \int_0^{n\bar{T}} (t-s)^{\beta-1} E_{\beta,\beta}\big(B(t-s)^\beta\big) k\big(s,x(s)\big)\,ds \\
&\quad + \int_{n\bar{T}}^t (t-s)^{\beta-1} E_{\beta,\beta}\big(B(t-s)^\beta\big) k\big(s,x(s)\big)\,ds \\
&\le \bar{y}(t) + \int_{n\bar{T}}^t (t-s)^{\beta-1} E_{\beta,\beta}\big(B(t-s)^\beta\big) k\big(s,x(s)\big)\,ds, \quad t \in \big[n\bar{T}, (n+1)\bar{T}\big],
\end{aligned}$$

where

$$\bar{y}(t) = y(t) + \int_0^{n\bar{T}} (t-s)^{\beta-1} E_{\beta,\beta}\bigl(B(t-s)^\beta\bigr) k\bigl(s,u(s)\bigr)\,ds.$$

Finally, defining $\mathcal{G}^{(n)} : C([n\bar{T},(n+1)\bar{T}]) \to C([n\bar{T},(n+1)\bar{T}])$ by

$$\bigl(\mathcal{G}^{(n)}z\bigr)(t) = \bar{y}(t) + \int_{n\bar{T}}^t (t-s)^{\beta-1} E_{\beta,\beta}\bigl(B(t-s)^\beta\bigr) k\bigl(s,z(s)\bigr)\,ds,$$

and using the fact that equation (5) has a unique solution due to hypothesis (ii), we can proceed as in the first part of this proof to see that $x \le u$ on $[0,(n+1)\bar{T}]$. Thus, the result follows using induction on n. □

3 A class of nonlinear fractional-order systems

In this section we establish two sufficient conditions for the stability of a deterministic semilinear Volterra integral equation. Thus, we improve the results in [45] for this kind of systems when the noise is null (*i.e.*, $\mathcal{Z}$ in (1) is equal to zero).

3.1 A constant as initial condition

This part is devoted to a refinement of Theorem 1 of [24] in the one-dimensional case. Toward this end, in this section, we suppose that the initial condition is a constant. That is, we first consider the fractional equation

$$\begin{aligned} X(t) &= x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} A X(s)\,ds \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h\bigl(X(s)\bigr)\,ds, \quad t \ge 0, \end{aligned} \tag{6}$$

with $x_0 \in \mathbb{R}$, $\beta \in (0,1)$, $A < 0$, and $h : \mathbb{R} \to \mathbb{R}$ a measurable function.

In the remaining of this paper we deal with the following hypotheses.

(H1) There is a constant $C > 0$ such that $A + C < 0$ and $|h(x)| \le C|x|$, for all $x \in \mathbb{R}$.

(H2) There are $\delta_0 > 0$ and $C > 0$ such that $A + C < 0$ and $|h(x)| \le C|x|$, for $|x| < \delta_0$.

Now, we consider several definitions of stability.

Definition 1 Any solution X to equation (6) is said to be:

(i) globally stable in the large if $X(t)$ goes to zero as t tends to infinity, for all $x_0 \in \mathbb{R}$;

(ii) Mittag-Leffler stable if there is $\delta > 0$ such that $|x_0| < \delta$ implies

$$\bigl|X(t)\bigr| \le \bigl[m(x_0) E_{\beta,1}\bigl(Bt^\beta\bigr)\bigr]^b, \quad t \ge 0,$$

where $\beta \in (0,1)$, $B < 0$, $b > 0$, and m is a positive and locally Lipschitz function with $m(0) = 0$;

(iii) stable if for $\varepsilon > 0$, there is $\delta > 0$ such that $|x_0| < \delta$ implies $|X(t)| < \varepsilon$, for all $t \ge 0$;

(iv) stable in the large if there is $\delta > 0$ such that $|x_0| < \delta$ implies $\lim_{t\to\infty} X(t) = 0$;

(v) asymptotically stable if it is stable and stable in the large.

Remark 1 Observe that, under the assumptions that h is continuous and satisfies (H1), equation (6) has at least one solution on $[0,\infty)$ because of [18] (Theorems 3.1 and 4.2). Indeed, in [18] (Theorem 4.2) we can consider

$$g(t,x)=\begin{cases}0 & \text{if } x\le 0,\\ (|A|+C)x & \text{if } x>0.\end{cases}$$

Similarly, for a continuous function h satisfying (H2), we introduce the function

$$\varphi(x)=\begin{cases}x & \text{if } |x|\le \delta_0/2,\\ \delta_0 & \text{if } x>\delta_0/2,\\ -\delta_0 & \text{if } x<-\delta_0/2.\end{cases}$$

Then, using [18] (Theorems 3.1 and 4.2) again, the equation

$$X(t)=x_0+\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\bigl(AX(s)+h\bigl(\varphi\bigl(X(s)\bigr)\bigr)\bigr)\,ds \tag{7}$$

has at least one solution defined on $[0,\infty)$ due to $|Ax+h(\varphi(x))|\le|Ax|+C|\varphi(x)|\le(|A|+C)|x|$. Hence equation (6) has at least one continuous solution on $[0,\infty)$ if (7) is stable and x_0 is small enough because, in this case, the solution of (7) is also a solution of equation (6) and $h\circ\varphi$ is bounded. So, without loss of generality we can assume that (6) has at least one continuous solution because one of the main purposes of the paper is to deal with the stability of (1).

Our first stability result for any continuous solution of equation (6) is the following.

Proposition 1 *Assume either* (H2) *or* (H1) *is satisfied. Then any continuous solution X to equation* (6) *is stable.*

Proof Let (H2) (resp. (H1)) be true and $x_0\in(0,\delta_0)$ (resp. $x_0>0$). Then the continuity of X implies that there is $\tau>0$ such that $X(t)\in(0,\delta_0)$ (resp. $X(t)>0$) for all $t\in[0,\tau]$. Consequently

$$0<X(t)\le x_0+\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}[A+C]X(s)\,ds<x_0,\quad t\in[0,\tau]. \tag{8}$$

In other words, (8) shows that X is less than x_0 if $X>0$ on $[0,\tau]$. Thus, we can proceed as in [50] (Lemma 2) using (H2) (resp. (H1)) to see that $X(t)>0$ for all $t\ge 0$, which implies that X is stable.

Finally, for $x_0<0$ and X a solution of (6), we see that $-X$ is a solution of

$$Y(t)=-x_0+\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\bigl[AY(s)\,ds+\hat{h}\bigl(Y(s)\bigr)\bigr]\,ds,\quad t\ge 0,$$

with $\hat{h}(x)=-h(-x)$. Therefore the proposition follows. □

Now we establish the main result of this subsection.

Proposition 2 *Let h be a function satisfying* (H2) (*resp.* (H1)). *Then any continuous solution of equation* (6) *is Mittag-Leffler stable and therefore is also asymptotically stable* (*resp. globally stable in the large*).

Proof Let (H2) (resp. (H1)) be satisfied and $0 < x_0 < \delta_0$ (resp. $x_0 > 0$). Then $0 < X(t) < \delta_0$ (resp. $X(t) > 0$) by [22] (see (8)).

On the one hand, consider the solution Z of the following linear fractional equation:

$$Z(t) = 2x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}[A+C]Z(s)\,ds, \quad t \geq 0.$$

Then by the continuity of the solutions X and Z, there exists $\tau > 0$ such that, for all $t \in (0,\tau)$, we have $0 < X(t) < Z(t)$. If this inequality is satisfied for any $t > 0$, we can ensure that X is asymptotically stable (resp. and globally stable in the large), and that this solution is also Mittag-Leffler stable because the solution Z of last equation is given by (see [23] or (3))

$$Z(t) = 2x_0 E_{\beta,1}\bigl([A+C]t^\beta\bigr), \quad t \geq 0.$$

We now suppose that there exists $t_0 > 0$ such that $X(t_0) = Z(t_0)$ and $X(t) < Z(t)$, for $t < t_0$. Set $Y = X - Z$, then

$$\begin{aligned} Y(t) = {}& -x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} AY(s)\,ds \\ & + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}\bigl[h\bigl(X(s)\bigr) - CZ(s)\bigr]\,ds, \quad t \geq 0. \end{aligned}$$

From (3) (see also [23]) we observe that Y also satisfies the equality

$$Y(t) = -x_0 E_{\beta,1}\bigl(At^\beta\bigr) + \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\bigl(A(t-s)^\beta\bigr)\bigl[h\bigl(X(s)\bigr) - CZ(s)\bigr]\,ds, \quad t \geq 0.$$

For $s \in (0,t_0)$, we have $|h(X(s))| \leq CX(s) < CZ(s)$. Thus $h(X(s)) - CZ(s) < 0$. Consequently, by the completely monotonic property of $E_{\beta,\beta}$ (*e.g.*, see Miller *et al.* [48] or Schneider [49]), we have $Y(t_0) < 0$, and this is a contradiction because it is supposed that $Y(t_0) = 0$. Now we can conclude that X is Mittag-Leffler stable.

Finally we consider the case that $-\delta_0 < x_0 < 0$ (resp. $x_0 < 0$). Note that $\hat{X} = -X$ is such that

$$\hat{X}(t) = -x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} A\hat{X}(s)\,ds + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}\tilde{h}\bigl(\hat{X}(s)\bigr)\,ds, \quad t \geq 0,$$

with $\tilde{h}(x) = -h(-x)$. Hence, by the first part of this proof and the fact that $\tilde{h}$ satisfies (H2) (resp. (H1)), we see that the proof is complete. □

Remark Let X be a solution to equation (6). Wen *et al.* [24] (Theorem 1) have proved that the solution to equation (6) is stable if $\lim_{|x|\to 0} \frac{|h(x)|}{|x|} \to 0$. Also, Zhang and Li [51] have used an equality similar to (3) to prove that X is asymptotically stable for the case that $\lim_{x\to 0} \frac{|h(x)|}{|x|} = 0$, $\beta \in (1,2)$ and $\beta + \frac{1}{|A|} < 2$. Proposition 2 establishes that X is asymptotically stable under a weaker condition. Namely (H2). This is possible because we use a comparison type result and the fact that this solution does not change sign.

3.2 A function as initial condition

Here we treat the case that the initial condition is a function satisfying some suitable conditions.

Consider the following deterministic Volterra integral equation:

$$\begin{aligned} X(t) &= \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} A X(s)\,ds \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h\bigl(X(s)\bigr)\,ds, \quad t \geq 0. \end{aligned} \tag{9}$$

Here $\beta \in (0,1)$, $A < 0$, and $h : \mathbb{R} \longmapsto \mathbb{R}$ and $\xi : \mathbb{R}^+ \longmapsto \mathbb{R}$ are two measurable functions.

Concerning the existence of a continuous solution of equation (9) we remark the following. For a continuous function h as in (H1) and ξ continuous, we can consider the equation

$$Z(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f\bigl(s, Z(s)\bigr)\,ds,$$

where $f(s,x) = A(x+\xi_s) + h(x+\xi_s)$, which has a solution Z due to Theorem 4.2 in [18] (with $g(s,x) = (|A|+C)(x+|\xi_s|)$) and (3). Therefore $Z+\xi$ is a solution of (9). Similarly if ξ is 'small enough' and h is either a continuous Lipschitz function on a neighborhood of zero, or as in (H2), then we can proceed as in Remark 1 to see that (9) has at least one solution in this case. Therefore, as in Remark 1, we can assume that (9) has at least one continuous solution.

On the other hand, in this paper we analyze several stability criteria for different classes $\mathcal{E}$ of initial conditions. Sometimes $\mathcal{E}$ is a subset of a normed linear space $\mathcal{X}$ of continuous functions endowed with the norm $\|\cdot\|_{\mathcal{X}}$. In other words we consider normed linear spaces $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$. Mainly, in the remaining of this paper, we deal with the following classes of initial conditions.

Definition 2 We have the following assumptions on ξ:

1. If the initial condition ξ is continuous on $[0,\infty)$ and we have $\xi_\infty \in \mathbb{R}$ such that, given $\varepsilon > 0$, there exists $t_0 > 0$ such that $|\xi_s - \xi_\infty| \leq \varepsilon$ for any $s \geq t_0$, we say that ξ belongs to the family $\mathcal{E}^1$.
2. $\mathcal{E}^2$ is the set of all functions ξ of class $C^1(\mathbb{R}_+)$ (*i.e.*, ξ has a continuous derivative on $\mathbb{R}_+$) such that

$$\lim_{t\to\infty} |\xi_t|/t^\beta = 0 \quad \text{and} \quad \bigl|\xi_t'\bigr| \leq \frac{\tilde{C}}{t^{1-\upsilon}}, \quad \text{for some } \upsilon \in (0,\beta) \text{ and } \tilde{C} \in \mathbb{R}.$$

3. $\mathcal{E}^3$ is the space of continuous functions of the form

$$\xi_t = \frac{1}{\Gamma(\eta)} \int_0^t (t-s)^{\eta-1} g(s)\,ds, \tag{10}$$

with $g \in L^1([0,\infty)) \cap L^p([0,\infty))$, $\eta \in (0, \beta+1)$, and $p > \frac{1}{\eta} \vee 1$.

The stability concepts that we develop in this section are the following.

Definition 3 Let $\mathcal{E} \subset \mathcal{X}$. A solution X of (9) is said to be:

(i) globally stable in the large for the class $\mathcal{E}$ (or globally $\mathcal{E}$-stable in the large) if $X(t)$ tends to zero as $t \to \infty$, for every $\xi \in \mathcal{E}$;

(ii) $\mathcal{E}$-stable if for $\varepsilon > 0$, there is $\delta > 0$ such that $\|X\|_{\infty,[0,\infty)} < \varepsilon$ for every $\xi \in \mathcal{E}$ satisfying $\|\xi\|_{\mathcal{X}} < \delta$;

(iii) asymptotically $\mathcal{E}$-stable if it is $\mathcal{E}$-stable and there is $\delta > 0$ such that $\lim_{t\to\infty} X(t) = 0$ for any $\xi \in \mathcal{E}$ such that $\|\xi\|_{\mathcal{X}} < \delta$.

In the following auxiliary result, $\mathcal{E}^4$ is the family of functions ξ having the form (10) with $\eta = \beta$ and g is a continuous function such that $\lim_{t\to\infty} g(t) = 0$. In this case, the involved norm is $\|\xi\|_{\mathcal{X}} = \|g\|_{\infty,[0,\infty)}$.

Lemma 2 *Let $B < 0$ and $\xi \in \mathcal{E}^4$. Then the solution to the equation*

$$Y(t) = \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} BY(s)\,ds, \quad t \geq 0,$$

is $\mathcal{E}^4$-stable and globally $\mathcal{E}^4$-stable in the large.

Proof We observe that, by (3), we have

$$\begin{aligned} Y(t) &= \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(B(t-s)^\beta\big) g(s)\,ds \\ &= \int_0^t s^{\beta-1} E_{\beta,\beta}\big(Bs^\beta\big) g(t-s)\,ds, \quad t \geq 0. \end{aligned}$$

So, the completely monotone property of $E_{\beta,\beta}$ (see Miller *et al.* [48] or Schneider [49]), and [4] (Theorem 1.6 and equality (1.99)) leads to

$$\begin{aligned} \big|Y(t)\big| &\leq \Big(\sup_{s\geq 0}\big|g(s)\big|\Big) \int_0^t s^{\beta-1} E_{\beta,\beta}\big(Bs^\beta\big)\,ds \\ &= \Big(\sup_{s\geq 0}\big|g(s)\big|\Big) t^\beta E_{\beta,\beta+1}\big(Bt^\beta\big) \\ &\leq \frac{C_{\beta,\beta+1}}{|B|} \|g\|_{\infty,[0,\infty)}. \end{aligned}$$

Thus, Y is $\mathcal{E}^4$-stable.

Also, by using (1.99) in [4] again, we are able to write

$$\begin{aligned} Y(t) &= \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(B(t-s)^\beta\big) g(s)\,ds \\ &= g(t) t^\beta E_{\beta,\beta+1}\big(Bt^\beta\big) \\ &\quad + \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(B(t-s)^\beta\big)\big[g(s) - g(t)\big]\,ds, \quad t \geq 0. \end{aligned}$$

Therefore, using Theorem 1.6 in [4] and the proof of Proposition 3.3.1 in [45] again, together with the facts that $B < 0$ and g is a continuous function such that $\lim_{t\to\infty} g(t) = 0$, we obtain $Y(t) \to 0$ as $t \to \infty$. □

Now we give a general result.

Theorem 1 *Let* (H2) (*resp.* (H1)) *be true, and* $\mathcal{E}$ *a family of continuous functions of a normed linear space* $\mathcal{X}$ *such that the solution of the equation*

$$Y(t) = \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} AY(s)\, ds, \quad t \geq 0, \tag{11}$$

is asymptotically $\mathcal{E}$*-stable* (*resp. globally* $\mathcal{E}$*-stable in the large*). *Then any continuous solution of equation* (9) *is also asymptotically* $\mathcal{E}$*-stable* (*resp. globally* $\mathcal{E}$*-stable in the large*).

Proof Suppose that (H1) (resp. (H2)) is true. Let X be a continuous solution to equation (9). Take $Z = X - Y$, then we have

$$Z(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \bigl[AZ(s) + h\bigl(X(s)\bigr)\bigr]\, ds, \quad t \geq 0.$$

Thus, (3) allows us to write

$$Z(t) = \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\bigl(A(t-s)^\beta\bigr) h\bigl(X(s)\bigr)\, ds, \quad t \geq 0.$$

Hence, for $\xi \in \mathcal{E}$ we have (resp. for $\xi \in \mathcal{E}$ such that $\|Y\|_{\infty,[0,\infty)} < \delta_0$, which gives $|\xi_0| = |Y(0)| < \delta_0$, the continuity of X implies that there is $t_0 > 0$ such that $\|X\|_{\infty,[0,t_0)} < \delta_0$ and therefore (H2) leads to)

$$\begin{aligned} \bigl|Z(t)\bigr| &\leq C \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\bigl(A(t-s)^\beta\bigr) \bigl|X(s)\bigr|\, ds \\ &\leq C \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\bigl(A(t-s)^\beta\bigr) \bigl|Z(s)\bigr|\, ds \\ &\quad + C \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\bigl(A(t-s)^\beta\bigr) \bigl|Y(s)\bigr|\, ds, \quad t \geq 0 \text{ (resp. } t \leq t_0), \end{aligned}$$

where we make use of the completely monotonic property of $E_{\beta,\beta}$. Invoking Lemma 1 and the uniqueness of the solutions for the involved equations, $|Z(t)| \leq u(t)$ for all $t \geq 0$ (resp. $t \leq t_0$), where u is the solution to

$$u(t) = \frac{C}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \bigl|Y(s)\bigr|\, ds + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [A+C] u(s)\, ds, \quad t \geq 0.$$

Finally observe that $|X(t)| \leq u(t) + |Y(t)|$ for $t \geq 0$ (resp. for $t \leq t_0$ such that $\|X\|_{\infty,[0,t_0)} < \delta_0$). Thus Lemma 2 implies that u is globally $\mathcal{E}^4$-stable in the large (resp. u is $\mathcal{E}^4$-stable and globally $\mathcal{E}^4$-stable in the large), which shows that the proof is complete. □

Remark For each $i \in \{1, \dots, n\}$ let $\mathcal{X}^i$ be a normed linear space of functions. Note that if $\xi = \sum_{i=1}^n \xi^{(i)}$, where $\xi^{(i)} \in \hat{\mathcal{E}}^i \subset \mathcal{X}^i$ and (11) is $\hat{\mathcal{E}}^{(i)}$-stable for each $i \in \{1, \dots, n\}$. Then (11) is also $\mathcal{E}$-stable, where $\mathcal{E}$ is the family of functions of the form $\sum_{i=1}^n \xi^{(i)}$ and the involved seminorm is $\|\xi\|_{\mathcal{X}} = \sum_{i=1}^n \|\xi^{(i)}\|_{\mathcal{X}^i}$. Indeed, by (3) we see that the solution Y is given by

$$Y(t) = \sum_{i=1}^n Y^{(i)}(t) = \sum_{i=1}^n \left(\xi_t^{(i)} + A \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\bigl(A(t-s)^\beta\bigr) \xi_s^{(i)}\, ds \right), \quad t \geq 0,$$

where, for each $i \in \{1, \ldots, n\}$, $Y^{(i)}$ is the unique solution to the linear equation

$$Y^{(i)}(t) = \xi_t^{(i)} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} A Y^{(i)}(s)\, ds, \quad t \geq 0.$$

In the following result we see that the family $\mathcal{E} := \{\xi \in C([0,\infty)) : \xi = \sum_{i=1}^3 \xi^{(i)}, \xi^{(i)} \in \mathcal{E}^i\}$ is an example of a family of functions for which the assumptions of Theorem 1 is satisfied. Here, $\|\cdot\|_{\mathcal{X}^1} = \|\cdot\|_{\infty,[0,\infty)}$, $\|\xi^{(2)}\|_{\mathcal{X}^2} = \|\xi^{(2)}_{\cdot} E_{\beta,1}(A\cdot^{\beta})\|_{\infty,[0,\infty)} + \|\cdot^{1-\upsilon}\xi^{(2)\prime}_{\cdot}\|_{\infty,[0,\infty)}$ and $\|\xi^{(3)}\|_{\mathcal{X}^3} = \|g\|_{L^1([0,\infty))} + \|g\|_{L^p([0,\infty))}$, where $\cdot^{1-\upsilon}\xi^{(2)\prime}_{\cdot}$ denotes $s \mapsto s^{1-\upsilon}\xi^{(2)\prime}_s$ and $\xi^{(3)}$ is given by the right-hand side of (10). Thus, in this case $\|\xi\|_{\mathcal{X}} = \sum_{i=1}^3 \|\xi^{(i)}\|_{\mathcal{X}^i}$.

Proposition 3 *Let $A < 0$ and $\beta \in (0,1)$. Then any solution to* (11) *is $\mathcal{E}$-stable and $\mathcal{E}$-stable in the large.*

Proof By the previous remark we only need that equation (11) is $\mathcal{E}^i$-stable and $\mathcal{E}^i$-stable in the large, for $i = 1,2,3$. To prove this, let Y be the solution to equation (11). The global $\mathcal{E}^i$-stability in the large has already been considered in [45] (Theorem 3.3). Now we divide the proof in three steps.

Step 1. Here we consider the case $i = 1$. Then (3) and (1.99) in [4] give, for $t \geq 0$,

$$\begin{aligned} |Y(t)| &\leq |\xi_t^{(1)}| + |A| \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(A(t-s)^\beta\big) |\xi_s^{(1)}|\, ds \\ &\leq \|\xi^{(1)}\|_{\infty,[0,\infty)} \left(1 + |A| \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(A(t-s)^\beta\big)\, ds\right) \\ &= \|\xi^{(1)}\|_{\infty,[0,\infty)} \big(1 + |A| t^\beta E_{\beta,\beta+1}\big(At^\beta\big)\big) \\ &\leq \|\xi^{(1)}\|_{\infty,[0,\infty)} (1 + C_{\beta,\beta+1}), \end{aligned}$$

which implies that the solution of (11) is $\xi^{(1)}$-stable.

Step 2. For $i = 2$, we get

$$\begin{aligned} |Y(t)| &\leq |\xi_t^{(2)} E_{\beta,1}\big(At^\beta\big)| + \left|A \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(A(t-s)^\beta\big)\big(\xi_s^{(2)} - \xi_t^{(2)}\big)\, ds\right| \\ &\leq \|\xi^{(2)}\|_{\mathcal{X}^{(2)}} \left(1 + |A| \int_0^t (t-s)^\beta E_{\beta,\beta}\big(A(t-s)^\beta\big) s^{\upsilon-1}\, ds\right), \quad t \geq 0. \end{aligned}$$

Consequently, [45] (the proof of Theorem 3.2.2) yields

$$\begin{aligned} |Y(t)| &\leq \|\xi^{(2)}\|_{\mathcal{X}^{(2)}} \big(1 + t^\upsilon \Gamma(\upsilon)\big[\upsilon E_{\beta,\upsilon+1}\big(At^\beta\big) - E_{\beta,\upsilon}\big(At^\beta\big)\big]\big) \\ &\leq C\|\xi^{(2)}\|_{\mathcal{X}^{(2)}}, \quad t \geq 0, \end{aligned}$$

where $C > 0$ is a constant and we have utilized that $\upsilon < \beta$.

Step 3. Finally we consider the case $i = 3$. In this scenario, from (3), we obtain

$$\begin{aligned} |Y(t)| &= \left|\int_0^t (t-s)^{\eta-1} E_{\beta,\eta}\big(A(t-s)^\beta\big) g(s)\, ds\right| \\ &= \left|\int_0^t s^{\eta-1} E_{\beta,\eta}\big(As^\beta\big) g(t-s)\, ds\right| \end{aligned}$$

$$\le \left| \int_0^{t\wedge 1} s^{\eta-1} E_{\beta,\eta}\big(As^\beta\big) g(t-s)\, ds \right| + \left| \int_{t\wedge 1}^t s^{\eta-1} E_{\beta,\eta}\big(As^\beta\big) g(t-s)\, ds \right|$$
$$= I_1^{(3)}(t) + I_2^{(3)}(t), \quad t \ge 0.$$

For $I_1^{(3)}$ we can apply the Hölder inequality to write, for $q^{-1} = 1 - p^{-1}$ and $C > 0$,

$$I_1^{(3)}(t) \le C_{\beta,\eta} \left[\int_0^1 s^{q(\eta-1)}\, ds \right]^{1/q} \left[\int_0^{t\wedge 1} \big|g(t-s)\big|^p\, ds \right]^{1/p}$$
$$\le C \|g\|_{L^p([0,\infty))}, \quad t \ge 0,$$

and for $I_2^{(3)}$ we use the fact that $\eta - 1 - \beta < 0$. Thus

$$\big|I_2^{(3)}(t)\big| \le \frac{C_{\beta,\eta}}{|A|} \|g\|_{L^1([0,\infty))}. \qquad \square$$

Remark Observe that $\mathcal{E}^1$ contains the bounded variation functions on compact sets of $\mathbb{R}_+$ of the form $\xi = \xi^{(1)} - \xi^{(2)}$, where $\xi^{(1)}$ and $\xi^{(2)}$ are two non-decreasing and bounded functions on $\mathbb{R}_+$.

The following result is an immediate consequence of Theorem 1 and Proposition 3.

Theorem 2 *Suppose that* (H2) (*resp.* (H1)) *holds. Let* ξ *be as in Proposition* 3. *Then any continuous solution to* (9) *is asymptotically* $\mathcal{E}$*-stable* (*resp. globally* $\mathcal{E}$*-stable in the large*).

4 Semilinear integral equations with additive noise

In this section we consider the equation

$$X(t) = \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \big[AX(s) + h\big(X(s)\big)\big]\, ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, d\theta_s, \quad t \ge 0. \tag{12}$$

Here ξ, β, A, and h are as in equation (9). Henceforth we assume that $\alpha \in (1,2)$, $\theta = \{\theta_s, s \ge 0\}$ is a γ-Hölder continuous function with $\gamma \in (0,1)$ such that $\theta_0 = 0$ and $\gamma + \alpha > 2$, and f is a τ-Hölder continuous function in $C^1(\mathbb{R}_+)$, with $\tau + \gamma > 1$. Note that, in this case, the Young integral in the right-hand side of (12) is equal to $\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, d\tilde{\theta}_s$, where $\tilde{\theta}_s = \int_0^s f(r)\, d\theta_r$ due to [45] (Lemma 2.4). Thus, (3) is still true for (12) and [45] (Lemma 2.7) implies

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, d\theta_s = \frac{\alpha - 1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2} \tilde{\theta}_s\, ds.$$

Hence, the existence of a continuous solution to (12) can be considered as in Section 3.2.

Definition 4 Let $\mathcal{E} \subset \mathcal{X}$ be a family of continuous functions. We say that a solution X of (12) is:

(i) $(\mathcal{E}, p)$-stable if for $\varepsilon > 0$, there is $\delta > 0$ such that $\|X\|_{\infty,[0,\infty)} < \varepsilon$ for any (ξ, f, θ) such that

$$\|\xi\|_{\mathcal{X}} + \|f\theta\|_{L^1([0,\infty))} + \|f\theta\|_{L^p([0,\infty))} + \|\dot{f}\theta\|_{L^1([0,\infty))} < \delta; \tag{13}$$

(ii) asymptotically $(\mathcal{E},p)$-stable if it is $(\mathcal{E},p)$-stable and there is $\delta > 0$ such that $\lim_{t\to\infty} X(t) = 0$ for any (ξ, f, θ) satisfying (13).

An extension of Theorem 1 is the following.

Theorem 3 *Let* (H2) (*resp.* (H1)) *be satisfied and* $\mathcal{E}$ *a class of continuous functions such that the solution of the equation*

$$Y(t) = \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} AY(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, d\theta_s, \quad t \geq 0, \tag{14}$$

is asymptotically $(\mathcal{E},p)$*-stable* (*resp. globally* $\mathcal{E}$*-stable in the large*). *Then any continuous solution of* (12) *is also asymptotically* $(\mathcal{E},p)$*-stable* (*resp. globally* $\mathcal{E}$*-stable in the large*).

Proof Observe $X(0) = \xi_0$. Consequently the proof is similar to that of Theorem 1. □

Now we state a consequence of Theorem 3.

Theorem 4 *Assume* (H2) (*resp.* (H1)) *holds. Let* ξ *be as in Proposition* 3, $f \in C^1((0,\infty))$ *such that* $\dot{f}\theta \in L^1([0,\infty))$ *and* $f\theta \in L^1([0,\infty)) \cap L^p([0,\infty))$ *for some* $p > \frac{1}{\alpha-1}$, *and* $\beta + 1 > \alpha$. *Then any continuous solution to* (12) *is asymptotically* $(\mathcal{E},p)$*-stable* (*resp. globally* $\mathcal{E}$*-stable in the large*).

Proof Suppose that (H2) (resp. (H1)) is satisfied. By Theorem 3 we only need to see that the solution Y of equation (14) is asymptotically $(\mathcal{E},p)$-stable (resp. globally $\mathcal{E}$-stable in the large). Toward this end, we invoke (3) and [45] (Lemma 2.4) to establish

$$\begin{aligned} Y(t) &= \xi_t + A \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(A(t-s)^\beta\big)\xi_s\, ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\beta,\alpha}\big(A(t-s)^\beta\big) f(s)\, d\theta_s \\ &= I_1(t) + I_2(t) + I_3(t), \quad t \geq 0. \end{aligned}$$

Thus, considering Proposition 3 and [45] (the proof of Proposition 4.1) we only need to show that, given $\varepsilon > 0$, $\|I_3\|_{\infty,[0,\infty)} < \varepsilon$ if $\|\xi\|_{\mathcal{X}} + \|f\theta\|_{L^1([0,\infty))} + \|f\theta\|_{L^p([0,\infty))} + \|\dot{f}\theta\|_{L^1([0,\infty))}$ is small enough. For this purpose, we observe that (1.83) in [4] and [45] (Lemma 2.7) imply

$$\begin{aligned} I_3(t) &= \int_0^t (t-s)^{\alpha-2} E_{\beta,\alpha-1}\big(A(t-s)^\beta\big)\theta_s f(s)\, ds \\ &\quad - \int_0^t (t-s)^{\alpha-1} E_{\beta,\alpha}\big(A(t-s)^\beta\big)\theta_s \dot{f}(s)\, ds \\ &= I_{3,1}(t) + I_{3,2}(t), \quad t \geq 0. \end{aligned} \tag{15}$$

For $I_{3,1}$ we have, from [4] (Theorem 1.6) and $q^{-1} = 1 - p^{-1}$,

$$\begin{aligned} \big|I_{3,1}(t)\big| &\leq \int_0^{1\wedge t} s^{\alpha-2} \big|E_{\beta,\alpha-1}\big(As^\beta\big)\big| \big|\theta_{t-s} f(t-s)\big|\, ds \\ &\quad + \int_{1\wedge t}^t s^{\alpha-2} \big|E_{\beta,\alpha-1}\big(As^\beta\big)\big| \big|\theta_{t-s} f(t-s)\big|\, ds \end{aligned}$$

$$\begin{aligned}
&\leq C_{\beta,\alpha-1}\left(\int_0^1 s^{q(\alpha-2)}\,ds\right)^{1/q}\left(\int_0^{1\wedge t}\left|\theta_{t-s}f(t-s)\right|^p\,ds\right)^{1/p}\\
&\quad + C_{\beta,\alpha-1}\int_0^t\left|\theta_{t-s}f(t-s)\right|\,ds\\
&\leq C(\|\theta f\|_{L^p([0,\infty))}+\|\theta f\|_{L^1([0,\infty))}),\quad t\geq 0. \qquad (16)
\end{aligned}$$

Finally, using [4] (Theorem 1.6) again and the fact that $\beta+1>\alpha$,

$$\begin{aligned}
I_{3,2}(t) &\leq \int_0^{1\wedge t} s^{\alpha-1}\left|E_{\beta,\alpha}\left(As^{\beta}\right)\right|\left|\theta_{t-s}\dot{f}(t-s)\right|\,ds\\
&\quad + \int_{1\wedge t}^{t} s^{\alpha-1}\left|E_{\beta,\alpha}\left(As^{\beta}\right)\right|\left|\theta_{t-s}\dot{f}(t-s)\right|\,ds\\
&\leq C_{\beta,\alpha}\int_0^t\left|\theta_{t-s}\dot{f}(t-s)\right|\,ds+\frac{C_{\beta,\alpha}}{|A|}\int_0^t\left|\theta_{t-s}\dot{f}(t-s)\right|\,ds\\
&\leq C\int_0^{\infty}\left|\theta_s\dot{f}(s)\right|\,ds,\quad t\geq 0.
\end{aligned}$$

Hence (15) and (16) show that the proof is complete. □

Observe that, in the previous proof, the inequality

$$I_{3,2}(t)\leq C\int_0^{\infty}\left|\theta_s\dot{f}(s)\right|\,ds,\quad t\geq 0,$$

is still true for $\beta+1\geq\alpha$, which is used in the proof of Theorem 6 below.

4.1 Stochastic integral equations with additive noise

In the remaining of this paper we suppose that all the introduced random variables are defined on a complete probability space $(\Omega,\mathcal{F},P)$.

Remark 2 Note that, in equation (12), we can consider a random variable $A:\Omega\to(-\infty,0)$, stochastic processes ξ, θ, and f, and a random field h such that for almost all ω, $A(\omega)$, $\xi_\cdot(\omega)$, $\theta_\cdot(\omega)$, $f(\omega,\cdot)$ and $h(\omega,\cdot)$ satisfy the hypotheses of Theorem 4 (or Theorem 3), then we can analyze stability for equation (12) ω by ω (*i.e.*, with probability one). An example for the process θ is a fractional Brownian motion B^H with Hurst parameter $H\in(0,1)$. Fractional Brownian motion is a centered Gaussian process with covariance

$$R_H(s,t)=\mathbf{E}\left(B_s^H B_t^H\right)=\frac{1}{2}\left(s^{2H}+t^{2H}-|t-s|^{2H}\right),\quad s,t\geq 0.$$

It is well known that B^H has γ-Hölder continuous paths on compact sets, for any exponent $\gamma<H$, due to the Kolmogorov continuity theorem (see Decreusefond and Üstünel [52]).

The last remark motivates the following.

Definition 5 A continuous solution X to equation (12) is said to be globally $\mathcal{E}$-stable in the mean if $\mathbf{E}|X(t)|\to 0$ as $t\to\infty$ for any process $\xi\in\mathcal{E}$.

An immediate consequence of the proof of Theorem 1, we can state the following extension of Theorem 3.

Theorem 5 *Let h satisfy* (H1), $A < 0$, $\mathcal{E}$ *a family of continuous processes and* f, θ *as in Remark* 2 *such that the solution to equation* (14) *is stable in the mean. Then any continuous solution to equation* (12) *is also* $\mathcal{E}$*-stable in the mean.*

Remark In [45] (Theorem 4.3) we can find examples of families of processes for which the solution of (14) is $\mathcal{E}$-stable in the mean.

Another definition motivated by Remark 2 is the following.

Definition 6 Let $\mathcal{E} \subset \mathcal{X}$ be a family of continuous functions. We say that a continuous process ξ belongs to $\mathcal{E}$ in the mean ($\xi \in \mathcal{E}_m$ for short) if $\mathbf{E}(|\xi|) \in \mathcal{E}$.

Now we consider the stochastic integral equation

$$\begin{aligned} X(t) = \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \big[AX(s) + h\big(X(s)\big)\big]\, ds \\ + \frac{1}{\Gamma(\beta+1)} \int_0^t (t-s)^{\beta} f(s)\, dB_s^{\gamma}, \quad t \geq 0. \end{aligned} \tag{17}$$

Here, in order to finish the paper, A, h, β, γ, and f are as in equation (12) such that $\beta + \gamma > 1$, and ξ is a continuous stochastic process. We remark that we interpret equation (17) path by path (*i.e.*, ω by ω).

The following definition is also inspired by Remark 2.

Definition 7 Let $\mathcal{E} \subset \mathcal{X}$ be a family of continuous functions. We say that a continuous solution to equation (17) is $(\mathcal{E}, p)$-stable in the mean if for a given $\varepsilon > 0$ there is $\delta > 0$ such that $\|\mathbf{E}|X|\|_{\infty,[0,\infty)} < \varepsilon$ for any $\xi \in \mathcal{E}_m$ such that

$$\big\|\mathbf{E}|\xi|\big\|_{\mathcal{X}} + \big\|f(\cdot)\cdot^{\gamma}\big\|_{L^1([0,\infty))} + \big\|f(\cdot)\cdot^{\gamma}\big\|_{L^p([t_0,\infty))} + \big\|\dot{f}(\cdot)\cdot^{\gamma}\big\|_{L^1([0,\infty))} < \delta.$$

Remark In this definition, if $\xi = \sum_{i=1}^n \xi^{(i)}$, with $\xi^{(i)} \in \mathcal{E}_m$, then we set $\|\xi\|_{\mathcal{X}} = \sum_{i=1}^n \|\xi^{(i)}\|_{\mathcal{X}}$.

Theorem 6 *Let* (H2) *be true,* ξ *as in Proposition* 3, $p > \frac{1}{\beta}$, *and* $f \in C^1((0,\infty))$ *a positive function with negative derivative such that* $(r \mapsto r^{\gamma}|\dot{f}(r)|) \in L^1([0,\infty))$ *and* $(r \mapsto r^{\gamma} f(r)) \in L^1([0,\infty)) \cap L^p([0,\infty))$. *Moreover, let h be a non-decreasing and locally Lipschitz function, which is concave on* $\mathbb{R}_+$ *and convex on* $\mathbb{R}_- \cup \{0\}$. *Then the solution to equation* (17) *is* $(\tilde{\mathcal{E}}, p)$*-stable in the mean, where* $\xi \in \tilde{\mathcal{E}}$ *if and only if* $\xi = \xi^{(1)} - \xi^{(2)}$ *with* ξ^1, ξ^2 *two non-negative, non-decreasing, and continuous processes in* $\mathcal{E}_m$.

Proof Let X be the continuous solution to equation (17). Then (3) implies

$$\begin{aligned} X(t) &= \xi_t E_{\beta,1}\big(At^{\beta}\big) + A \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(A(t-s)^{\beta}\big)(\xi_s - \xi_t)\, ds \\ &\quad + \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(A(t-s)^{\beta}\big) h\big(X(s)\big)\, ds \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t (t-s)^{\beta} E_{\beta,\beta+1}\big(A(t-s)^{\beta}\big) f(s)\, dB_s^{\gamma} \\
&\leq \xi_t^{(1)} E_{\beta,1}\big(At^{\beta}\big) + A\int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(A(t-s)^{\beta}\big)\big(\xi_s^{(1)}-\xi_t^{(1)}\big)\, ds \\
&+\int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(A(t-s)^{\beta}\big) h\big(X(s)\big)\, ds \\
&+\int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(A(t-s)^{\beta}\big)\big|B_s^{\gamma}\big| f(s)\, ds \\
&-\int_0^t (t-s)^{\beta} E_{\beta,\beta+1}\big(A(t-s)^{\beta}\big)\dot{f}(s)\big|B_s^{\gamma}\big|\, ds, \quad t \geq 0,
\end{aligned}
$$

where the last inequality follows from the facts that $0 \leq \xi^{(1)}, \xi^{(2)}$ are two non-decreasing processes, $f, (-\dot{f}) \geq 0$ and from [45] (Lemma 2.7). Therefore, we can state, by Lemma 1, that $X \leq X^{(1)}$ where $X^{(1)}$ is the solution to

$$
\begin{aligned}
X^{(1)}(t) &= \xi_t^{(1)} E_{\beta,1}\big(At^{\beta}\big) + A\int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(A(t-s)^{\beta}\big)\big(\xi_s^{(1)}-\xi_t^{(1)}\big)\, ds \\
&+\int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(A(t-s)^{\beta}\big) h\big(X^{(1)}(s)\big)\, ds \\
&+\int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(A(t-s)^{\beta}\big)\big|B_s^{\gamma}\big| f(s)\, ds \\
&-\int_0^t (t-s)^{\beta} E_{\beta,\beta+1}\big(A(t-s)^{\beta}\big)\dot{f}(s)\big|B_s^{\gamma}\big|\, ds, \quad t \geq 0.
\end{aligned} \tag{18}
$$

Observe that we also have $X^{(1)}(t) \geq 0$ due to $h(0) = 0$, Lemma 1, and

$$
-X^{(1)}(t) \leq \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(A(t-s)^{\beta}\big)\hat{h}\big(-X^{(1)}(s)\big)\, ds, \quad t \geq 0,
$$

with $\hat{h}(x) = -h(-x)$, $x \in \mathbb{R}$. Proceeding similarly we have $-X(t) \leq X^{(2)}(t)$, with $X^{(2)}(t) > 0$ and

$$
\begin{aligned}
X^{(2)}(t) &= \xi_t^{(2)} E_{\beta,1}\big(At^{\beta}\big) + A\int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(A(t-s)^{\beta}\big)\big(\xi_s^{(2)}-\xi_t^{(2)}\big)\, ds \\
&+\int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(A(t-s)^{\beta}\big)\hat{h}\big(X^{(2)}(s)\big)\, ds \\
&+\int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(A(t-s)^{\beta}\big)\big|B_s^{\gamma}\big| f(s)\, ds \\
&-\int_0^t (t-s)^{\beta} E_{\beta,\beta+1}\big(A(t-s)^{\beta}\big)\dot{f}(s)\big|B_s^{\gamma}\big|\, ds, \quad t \geq 0.
\end{aligned} \tag{19}
$$

In other words, we have

$$
\mathbf{E}\big(|X(t)|\big) \leq \mathbf{E}\big(X^{(1)}(t)\big) + \mathbf{E}\big(X^{(2)}(t)\big), \quad t \geq 0. \tag{20}
$$

Finally, observe that (18), (19), the fact that A is a negative number and the Jensen inequality give, for $\theta_s = s^\gamma$,

$$\begin{aligned}\mathbf{E}\big(X^{(1)}(t)\big) \le{}& \mathbf{E}\big(\xi_t^{(1)}\big)E_{\beta,1}\big(At^\beta\big)\\ &+ A\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}\big(A(t-s)^\beta\big)\mathbf{E}\big(\xi_s^{(1)} - \xi_t^{(1)}\big)\,ds\\ &+ \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}\big(A(t-s)^\beta\big)h\big(\mathbf{E}\big[X^{(1)}(s)\big]\big)\,ds\\ &+ \int_0^t (t-s)^{\beta}E_{\beta,\beta+1}\big(A(t-s)^\beta\big)f(s)\,d\theta_s, \quad t\ge 0\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}\big(X^{(2)}(t)\big) \le{}& \mathbf{E}\big(\xi_t^{(2)}\big)E_{\beta,1}\big(At^\beta\big)\\ &+ A\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}\big(A(t-s)^\beta\big)\mathbf{E}\big(\xi_s^{(2)} - \xi_t^{(2)}\big)\,ds\\ &+ \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}\big(A(t-s)^\beta\big)\hat{h}\big(\mathbf{E}\big[X^{(2)}(s)\big]\big)\,ds\\ &+ \int_0^t (t-s)^{\beta}E_{\beta,\beta+1}\big(A(t-s)^\beta\big)f(s)\,d\theta_s, \quad t\ge 0.\end{aligned}$$

Hence by (20), Lemma 1, hypothesis (H2), and the proofs of Proposition 3 and Theorem 4 we see that the result holds. Indeed, for $i = 1,2$,

$$\mathbf{E}\big(X^{(i)}(t)\big) \le u^{(i)}(t), \quad t\ge 0,$$

where $u^{(i)}$ is the unique solution to the equation

$$\begin{aligned}u^{(i)}(t) ={}& \mathbf{E}\big(\xi_t^{(i)}\big) + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}[A+C]u^{(i)}(s)\,ds\\ &+ \frac{1}{\Gamma(\beta+1)}\int_0^t (t-s)^\beta f(s)\,d\theta_s, \quad t\ge 0.\end{aligned}$$ □

Example 1 A function h that satisfies the conditions of Theorem 6 is

$$h(x) = \begin{cases} 1-e^{-Cx} & \text{if } x\ge 0,\\ e^{Cx}-1 & \text{if } x<0,\end{cases}$$

where $C > 0$. Indeed, we have

$$h'(x) = \begin{cases} Ce^{-Cx} & \text{if } x\ge 0,\\ Ce^{Cx} & \text{if } x<0.\end{cases}$$

Thus, given $\varepsilon > 0$ there is $\delta > 0$ such that

$$\big|h(x)\big| \le (C+\varepsilon)|x|, \quad \text{for } |x|\le\delta.$$

Example 2 Here we give a function that satisfies assumption 2 on Definition 2. Let $\xi_t = g(t)\sin\frac{1}{t}$, $t \geq 0$. The function g is bounded and satisfies $g(t) = \psi(t)c_0 t^{3-\upsilon} + \varphi(t)\frac{c_1}{1+t}$, where $\psi, \varphi \in C^\infty(\mathbb{R}_+)$ are such that

$$\psi(t) = \begin{cases} 1 & \text{if } t \in [0,1], \\ 0 & \text{if } t \geq 2 \end{cases} \quad \text{and} \quad \varphi(t) = \begin{cases} 0 & \text{if } t \in [0,1], \\ 1 & \text{if } t \geq 2. \end{cases}$$

Thus

$$\xi_t' = g'(t)\sin\frac{1}{t} - g(t)t^{-2}\cos\frac{1}{t}, \quad t \geq 0.$$

Now it is easy to verify our claim is true using straightforward calculations.

5 Conclusion

In this work we show that a useful tool to study several definitions of stability for some fractional equations is comparison results for fractional systems (see Lemma 1) and an equation in terms of the Mittag-Leffler functions (see representation (3)). Hence we can apply the properties of the Mittag-Leffler function to consider fractional systems with a function as initial condition and an additive noise, which is a Young integral that could be a functional of fractional Brownian motion.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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